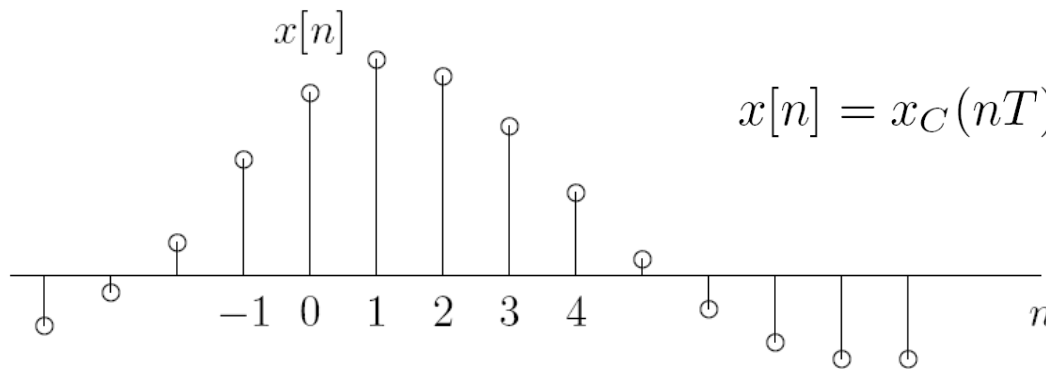
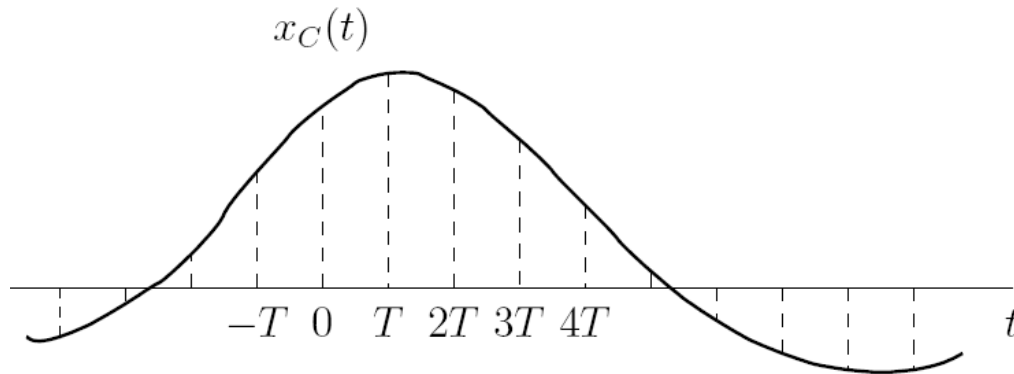


Amostragem

Amostragem



$$x[n] = x_C(nT), \quad -\infty < n < \infty$$

Qual é a relação entre as Transformadas de Fourier de $x_C(t)$ e $x[n]$?

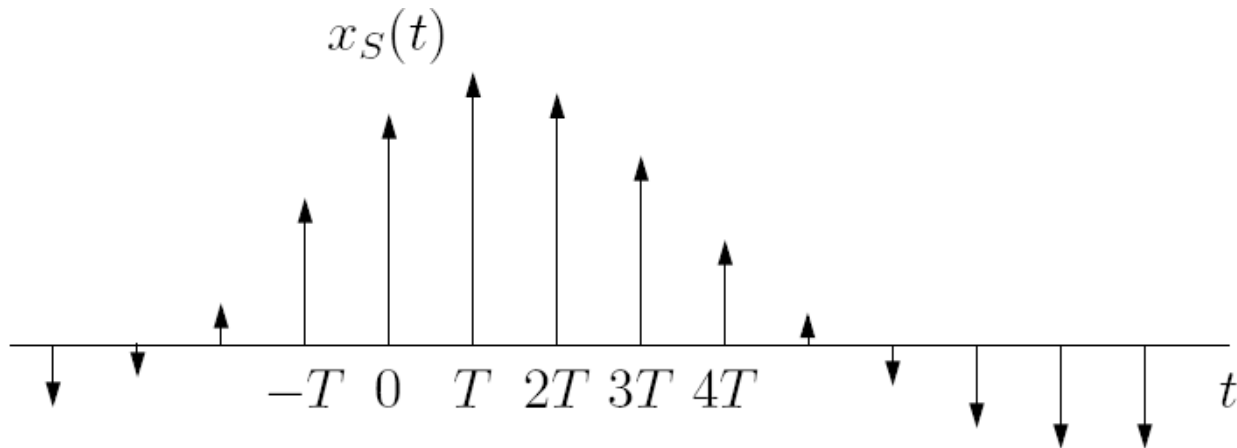
Amostragem

Modelo simples: amostragem impulsiva

$$x_C(t) \longrightarrow \otimes \longrightarrow x_S(t)$$

↑

$$\Delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Amostragem

$\Delta(t)$ pode ser expresso pela série de Fourier $\Delta(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$
cujos coeficientes são:

$$\begin{aligned}c_n &= \frac{1}{T} \int_{-T/2}^{T/2} \Delta(t) e^{-j\frac{2\pi n}{T}t} dt \\&= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT) e^{-j\frac{2\pi n}{T}t} dt \\&= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi n}{T}t} dt \\&= \frac{1}{T}\end{aligned}$$

Portanto: $\Delta(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{T}t}$

Amostragem

Podemos então escrever:

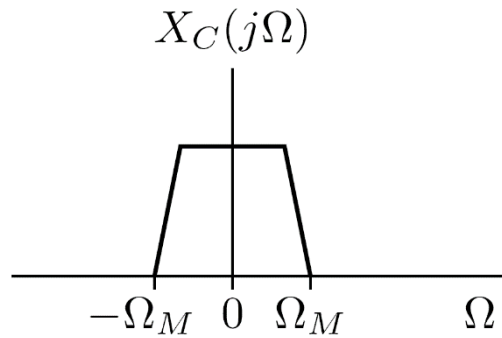
$$\begin{aligned}x_S(t) &= x_C(t)\Delta(t) \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} x_C(t)e^{j\frac{2\pi n}{T}t}\end{aligned}$$

cuja Transformada de Fourier (CTFT) é:

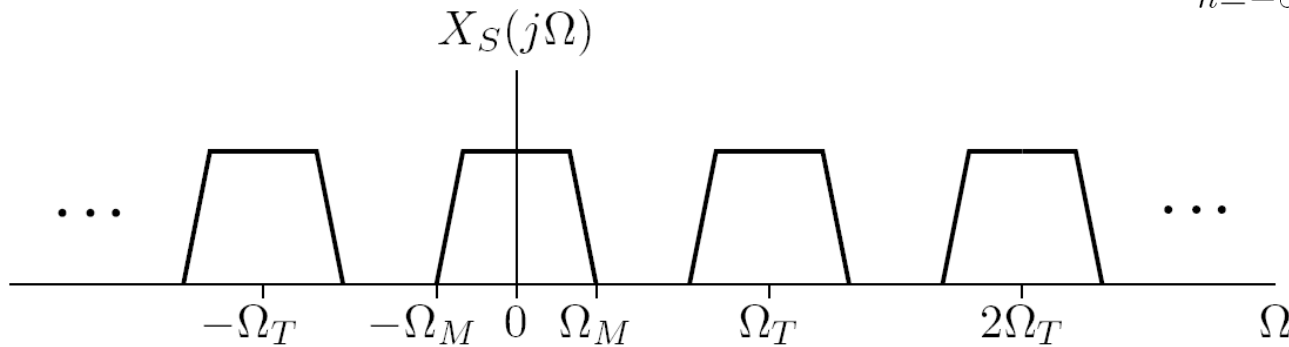
$$X_S(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C(j(\Omega - n\Omega_T))$$

$\Omega_T = 2\pi/T$ é a frequência de amostragem.

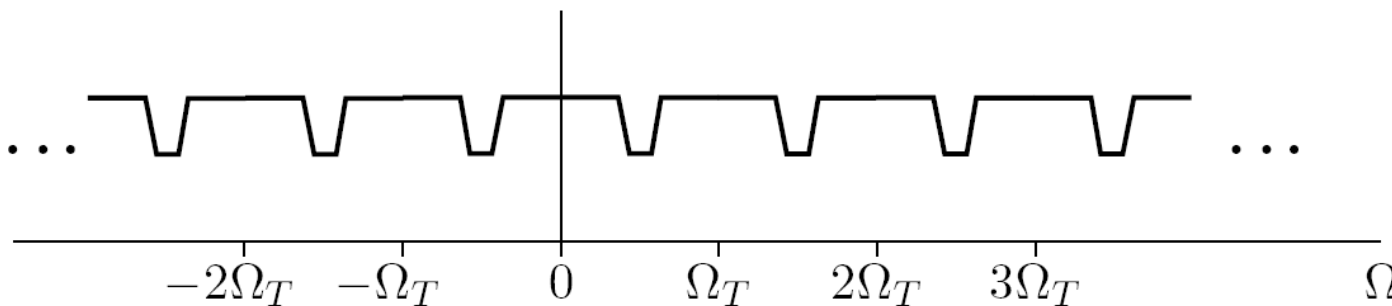
Amostragem



$$X_S(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C(j(\Omega - n\Omega_T))$$



$$\Omega_T > 2\Omega_M$$



$$\Omega_T < 2\Omega_M$$

\Rightarrow *aliasing*

Amostragem

Algumas definições:

Taxa de Nyquist: $\Omega_T = 2\Omega_M$ (amostragem crítica)

Oversampling: $\Omega_T > 2\Omega_M$

Undersampling: $\Omega_T < 2\Omega_M$

Amostragem

Relação entre $X_C(j\Omega)$ e $X(e^{j\omega})$:

$$\begin{aligned}x_S(t) &= x_C(t)\Delta(t) \\ &= \sum_{n=-\infty}^{\infty} x_C(nT)\delta(t - nT)\end{aligned}$$

Aplicando a **C**TFT:

$$\begin{aligned}X_S(j\Omega) &= \int_{-\infty}^{\infty} x_S(t)e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_C(nT)\delta(t - nT) \right) e^{-j\Omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\Omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT}\end{aligned}$$

Amostragem

CTFT de $x_S(t)$:
$$X_S(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT}$$

DTFT de $x[n]$:
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

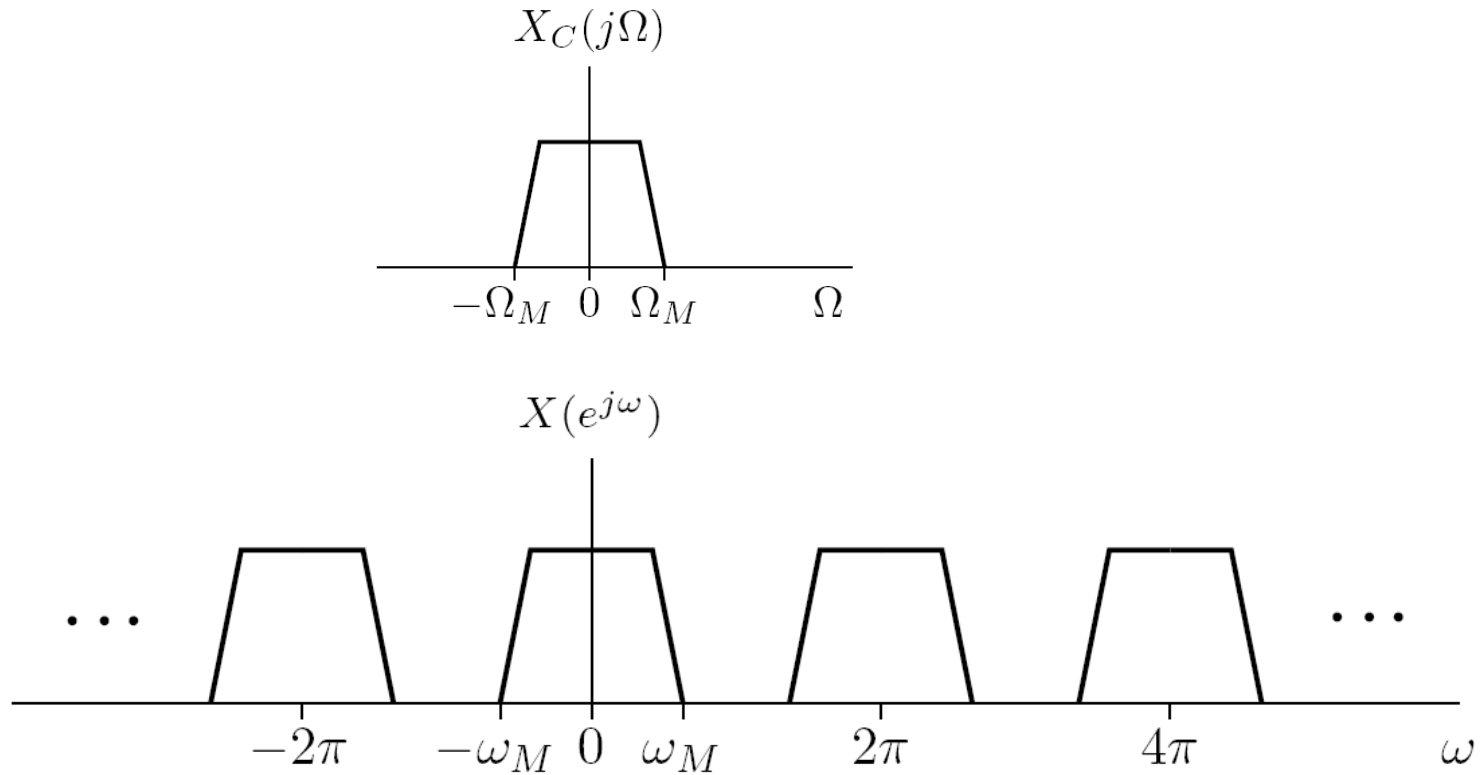
Portanto:
$$X_S(j\Omega) = X(e^{j\Omega T}) \quad \Rightarrow \quad \boxed{\omega = \Omega T}$$

$$X_S(j\omega/T) = X(e^{j\omega})$$

Como
$$X_S(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C(j(\Omega - n\Omega_T))$$

então, finalmente:
$$X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C \left(j \left(\frac{\omega - 2\pi n}{T} \right) \right)$$

Amostragem

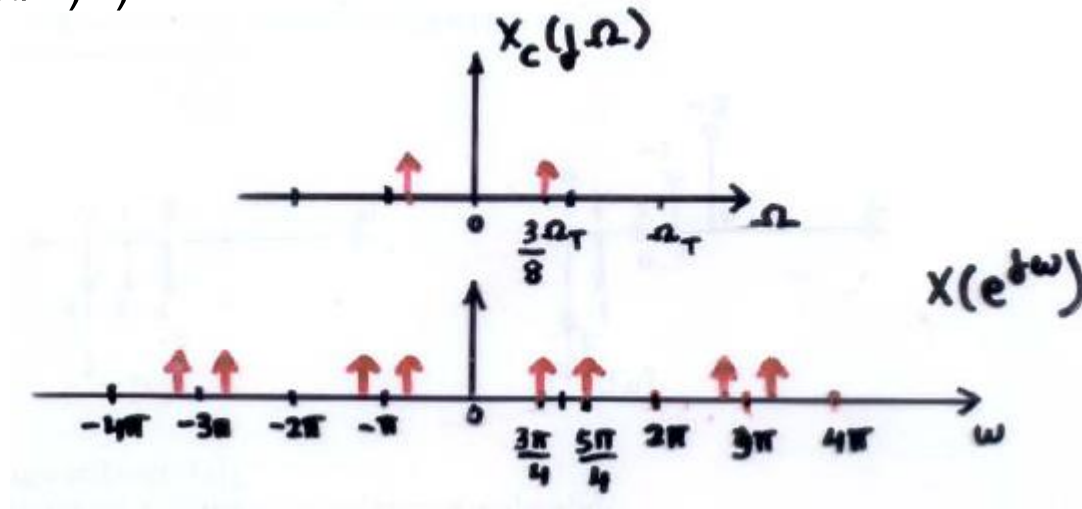


$$X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C \left(j \left(\frac{\omega - 2\pi n}{T} \right) \right)$$

Amostragem

$$x_C(t) = \cos\left(\left(\frac{3\Omega_T}{8}\right)t\right)$$

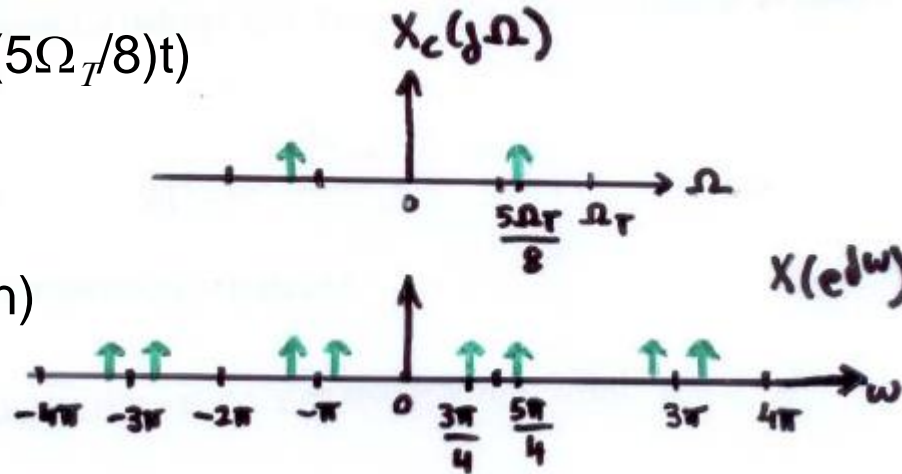
$$x[n] = \cos\left(\left(\frac{3\pi}{4}\right)n\right)$$



$$X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C\left(j\left(\frac{\omega - 2\pi n}{T}\right)\right)$$

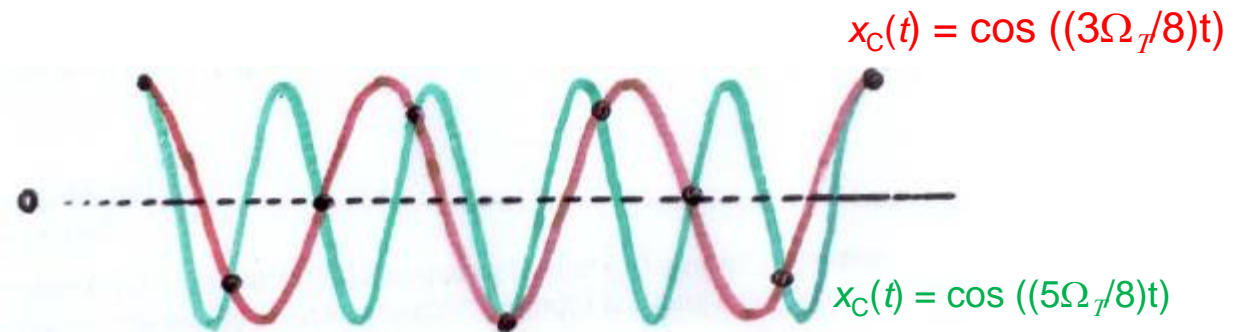
Amostragem

$$x_c(t) = \cos((5\Omega_T/8)t)$$



$$x[n] = \cos((3\pi/4)n)$$

No domínio do tempo:



Amostragem

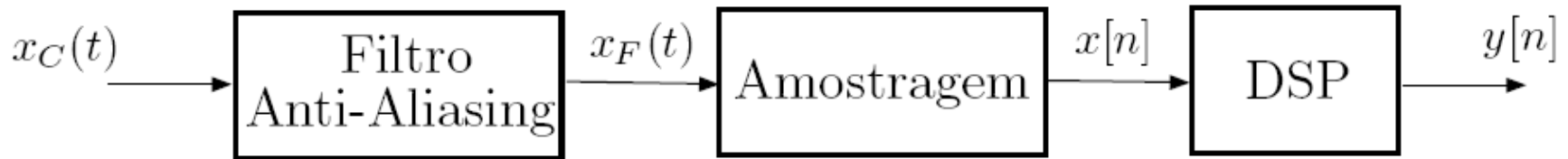
Teorema da Amostragem:

Um sinal contínuo $x_C(t)$ limitado em frequência, com $X_C(j\Omega) = 0$ para $|\Omega| > \Omega_M$, é unicamente determinado por suas amostras $x_C(nT)$ se

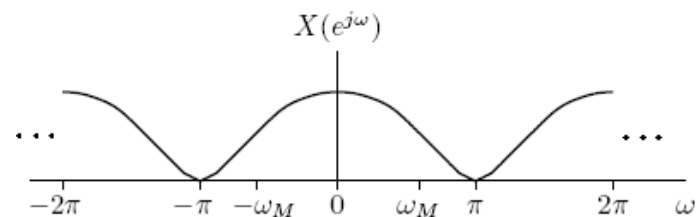
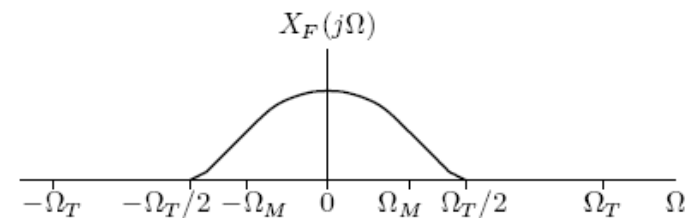
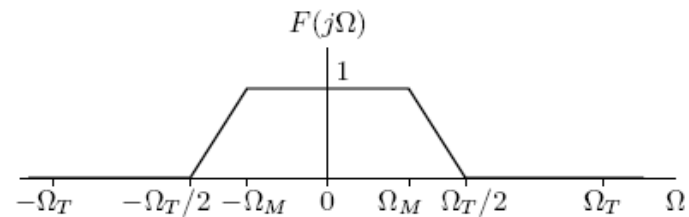
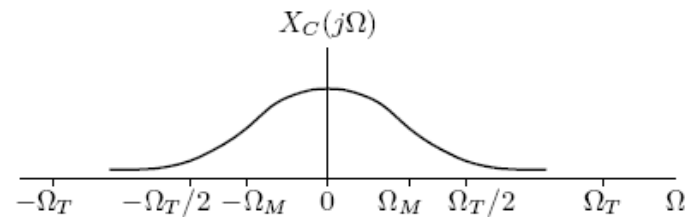
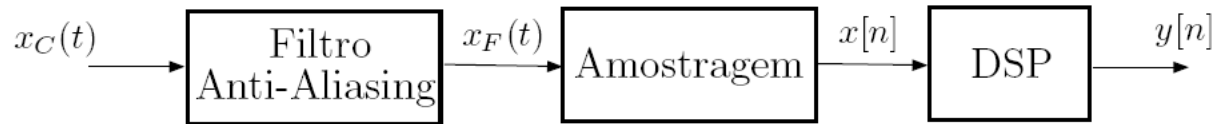
$$\Omega_T \geq 2\Omega_M$$

onde $\Omega_T = 2\pi/T$.

Geralmente o sinal de interesse possui energia espúria (ruído) em frequências elevadas \Rightarrow é necessário um filtro analógico para evitar *aliasing*.



Filtragem Anti-aliasing



Recuperação do Sinal Analógico

Se a condição de Nyquist for satisfeita, o sinal contínuo $x_c(t)$ pode ser recuperado, passando-se o trem de impulsos $x_s(t)$ por um filtro passa-baixas analógico $H_r(j\Omega)$ com frequência de corte Ω_C tal que

$$\Omega_M < \Omega_C < (\Omega_T - \Omega_M)$$

ou seja:

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_C \\ 0, & |\Omega| > \Omega_C \end{cases}$$

A resposta ao impulso de $H_r(j\Omega)$ é:

$$h_r(t) = \frac{T}{2\pi} \int_{-\Omega_C}^{\Omega_C} e^{j\Omega t} d\Omega = \frac{\text{sen}(\Omega_C t)}{\Omega_C t/2}$$

Recuperação do Sinal Analógico

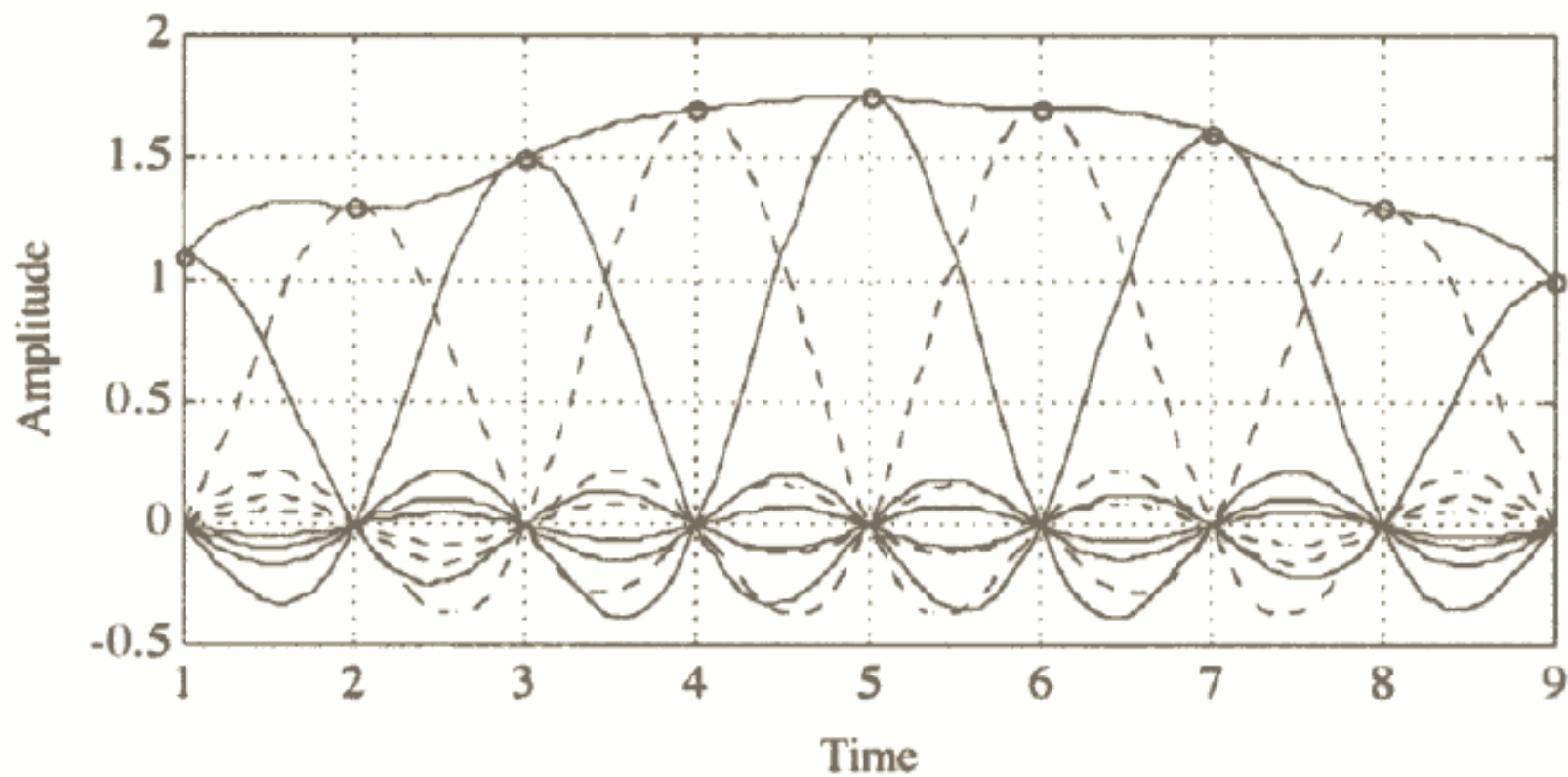
O trem de impulsos $x_S(t)$ é dado por

$$x_S(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT)$$

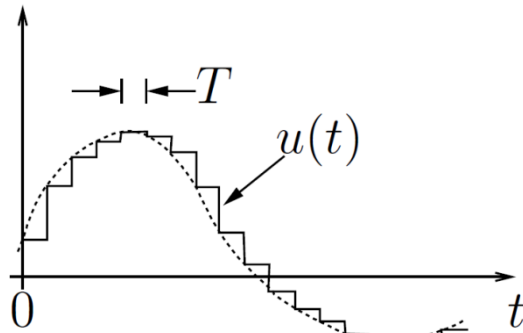
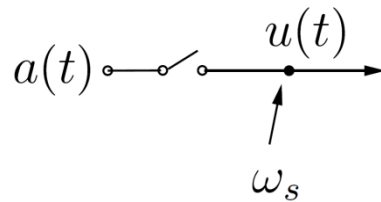
Portanto, a saída do filtro $H_r(j\Omega)$ com entrada $x_S(t)$, assumindo $\Omega_C = \Omega_T/2 = \pi/T$, é:

$$\begin{aligned}\hat{x}_C(t) &= \sum_{n=-\infty}^{\infty} x(n)h_r(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{\text{sen}(\pi(t - nT)/T)}{\pi(t - nT)/T}\end{aligned}$$

Recuperação do Sinal Analógico



Amostragem Sample-and-Hold



$$U(j\omega) = X(j\omega)S(j\omega)$$

$$X(j\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} A(j\omega - jk\omega_s)$$

$$S(j\omega) = \frac{2\pi}{\omega_s} \operatorname{sinc}\left(\frac{\omega}{\omega_s}\right) e^{-j\pi\omega/\omega_s}$$

